The Gradient of a Curve

In Rates of Change we estimated gradients of curves by drawing tangents. Differentiation computes them exactly.

Consider the chord joining (x, x^2) and a nearby point $\left(x+h, (x+h)^2\right)$ on $y = x^2$. As h shrinks, the chord becomes the tangent. Find the gradient of the chord, and decide what happens as $h \to 0$.

Definition. The derivative of y with respect to x is the gradient function, written

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \qquad f'(x), \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\ldots\right).$$

Theorem

$$y = kx^n \implies \frac{\mathrm{d}y}{\mathrm{d}x} = nkx^{n-1} \qquad \text{for any rational } n$$

Constants differentiate to 0, and sums differentiate term by term.

Example

Differentiate:

1. $y = x^7$
2. $y = 5x^4 - 3x^2 + 2x - 11$
3. $y = \frac{1}{2}x^6$

Textbook Exercises: SPS Course 6.1, Gradients Worksheet and Exercise 1

Differentiating Awkward Expressions

The rule only applies to *terms* of the form kx^n. Products, quotients and roots must be rewritten first.

Tip

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \frac{1}{x^3} = x^{-3} \qquad \frac{a + bx^2}{x} = ax^{-1} + bx \qquad (x+2)(x-5) = x^2 - 3x - 10$$

Example

Differentiate:

1. $y = \sqrt{x} + \dfrac{4}{\sqrt{x}}$
2. $y = 3x^2 + 6x^{\frac{1}{3}} + \dfrac{2x^3 - 7}{3\sqrt{x}}$

Example (Edexcel C1)

The curve C has equation

$$y = \frac{(x+3)(x-8)}{x}, \quad x > 0.$$

Find $\frac{dy}{dx}$ in its simplest form.

Example

Find the gradient of $y = 2x - 8\sqrt{x} + 5$ at $x = 4$, and find the point on the curve where the gradient is 0.

Textbook Exercises: SPS Course 6.1, Exercise 2A

Tangents and Normals

Fact — At the point $(a, f(a))$ on $y = f(x)$:

- the **tangent** has gradient $f'(a)$;
- the **normal** is perpendicular to the tangent: gradient $-\dfrac{1}{f'(a)}$.

Then use $y - y_1 = m(x - x_1)$.

Example (Edexcel C1)

The curve C has equation $y = x^3 - 2x^2 - x + 3$. The point $P(2, 1)$ lies on C.

1. Show that the equation of the tangent to C at P is $y = 3x - 5$.
2. The tangent to C at a point Q is parallel to the tangent at P. Find the coordinates of Q.

Example

Find the equation of the normal to $y = \sqrt{x}$ at the point $(9, 3)$, and the coordinates of the point where this normal meets the curve again.

Textbook Exercises: SPS Course 6.1, Exercise 3

Problems with Parameters

Example (Edexcel C1, adapted)

The curve C has equation $y = 2x^3 + kx^2 + 5x + 6$, where k is a constant.
The point P, where $x = -2$, lies on C. The tangent to C at P is parallel to the line $2y - 18x - 1 = 0$. Find

1. the value of k,
2. the y-coordinate of P,
3. the equation of the tangent at P in the form $ax + by + c = 0$.

Example

The curve $y = x^2 + ax + b$ has a tangent at $(2, 7)$ which passes through the origin. Find a and b.

Textbook Exercises: SPS Course 6.1, Exercise 2B

Differentiation and Kinematics

In Rates of Change, velocity and acceleration were gradients read from graphs. Now they are derivatives:

Fact —

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Example

The displacement, s metres, of a particle from a fixed point O after t seconds is

$$s = 24t^2 - t^3, \qquad 0 \le t \le 20.$$

1. Find expressions for the velocity and the acceleration.
2. Find the times at which the particle is at rest.
3. Find the maximum velocity of the particle.

Example

A ball is thrown vertically upwards so that its height above the ground after t seconds is $h = 2 + 15t - 5t^2$ metres.

1. Find the velocity when $t = 1$.
2. Find the greatest height reached.
3. Find the acceleration, and interpret it.

Textbook Exercises: SPS Course 6.1, Exercise 2A Q (kinematics parts) and Exercise 6